Math 152, Fall 2022
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WU # 8
Tuesday 9/27/22

Your name: ____________________________________

Names of people you worked with: ____________________________________

Task:

Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$. Use moment generating functions to show

$$\overline{X} \sim N(\mu, \sigma^2/n).$$

(Remember, the moment generating function for an arbitrary random variable W is $E[e^{Wt}]$, and many MGF formulations are given on the distribution sheet.)

Note: you are **not** proving the Central Limit Theorem here. Why not?

Solution:

By looking at the distribution sheet, we know:

$$\psi_{X_i}(t) = e^{\mu t + \sigma^2 t^2/2}.$$

The goal of this problem is to show that

$$\psi_{\overline{X}}(t) = e^{\mu t + (\sigma^2/n)t^2/2}.$$

To find the distribution of $\overline{X}$, we need to find the MGF of $\overline{X}$. Let $Y = \overline{X}$.

$$\begin{aligned}
\psi_{\overline{X}}(t) = \psi_Y(t) &= E[e^{Yt}] \\
&= E[e^{\overline{X}t}] \\
&= E[e^{\sum_i X_i \cdot t/n}] \\
&= E[\prod_i e^{X_i \cdot t/n}] \\
&= \prod_i E[e^{X_i \cdot t/n}] \\
&= \prod_i \psi_{X_i}(t/n) \\
&= \prod_i e^{\mu(t/n) + \sigma^2 (t/n)^2/2} \\
&= [e^{\mu(t/n) + \sigma^2 (t/n)^2/2}]^n \\
&= e^{n \cdot (\mu(t/n) + \sigma^2 (t/n)^2/2)} \\
&= e^{\mu t + (\sigma^2/n)t^2/2)}
\end{aligned}$$

which completes the proof.

The proof above is not the CLT for (at least) two reasons:

1. The result holds for any sample size n. That is, if the data start off as normal, their average will still be normal even if the sample size is very small (even if $n = 1$!).

2. The data described in this result are normal to start with. The CLT result is for any arbitrary distribution, but the result provided here applies only to data which come from a normal distribution.