Math 152, Fall 2022
Jo Hardin
WU # 7
Thursday 9/22/22

Your name: ____

Names of people you worked with: ____

Task:
Consider $Z \sim N(0,1), Y = Z^2$. The goal is to find the distribution of Y given the information that Z has a standard normal distribution.

Let Φ and ϕ be the cdf and pdf of Z.
Let F_Y and f_Y be the cdf and pdf of Y.

Find $F_Y(y)$ in terms of Φ. (Bonus: then take the derivative to find $f_y(y)$ in terms of ϕ.
Hint, start this problem in the following way:

$$F_Y(y) = P(Y \leq y) = \ldots$$

In the very next step you should plug in Z and keep going!

Solution:
If $Z \sim N(0,1), Y = Z^2$, then $Y \sim \chi^2_1$.

$$\begin{aligned}
F_Y(y) &= P(Y \leq y) = P(Z^2 \leq y) \\
&= P(-y^{1/2} \leq Z \leq y^{1/2}) \\
&= \Phi(y^{1/2}) - \Phi(-y^{1/2}) \qquad y > 0
\end{aligned}$$

$$\begin{aligned}
f_Y(y) &= \frac{\partial F_Y(y)}{\partial y} = \phi(y^{1/2}) \cdot \frac{1}{2} y^{-1/2} - \phi(-y^{1/2}) \cdot \frac{1}{2} - y^{-1/2} \\
&= \frac{1}{2} y^{-1/2} (\phi(y^{1/2}) + \phi(-y^{1/2})) \qquad y > 0 \\
\text{we know} \quad & \phi(y^{1/2}) = \phi(-y^{1/2}) = \frac{1}{\sqrt{2\pi}} e^{-y/2} \\
f_Y(y) &= y^{-1/2} \frac{1}{\sqrt{2\pi}} e^{-y/2} \qquad y > 0 \\
&= \frac{1}{2^{1/2}\pi^{1/2}} y^{1/2-1} e^{-y/2} \qquad y > 0 \\
Y &\sim \chi^2_1
\end{aligned}$$

note: $\Gamma(1/2) = \sqrt{\pi}$.