Your name: ____

Names of people you worked with: ____

Task: Let θ denote the average number of defects per 100 feet of tape. Assume $X_1, X_2, \ldots X_{12} \sim \text{Poisson}(\theta)$.

θ is unknown, but the prior on θ is a gamma distribution with $E[\theta] = \alpha/\beta = 2/10, \alpha = 2, \beta = 10$. When a 1200 foot roll of tape is inspected, exactly 4 defects are found.

In the interest of time, I've written out both the prior and the likelihood. You should be able to come up with these (relatively quickly) on your own.

Find the posterior distribution of $\theta|\underline{x}$. Completely specify the distribution.

$$\begin{aligned} \text{Prior: } \xi(\theta) &= \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta} = \frac{10^2}{\Gamma(2)}\theta e^{-10\theta} \\ \text{Likelihood: } f(\underline{x}|\theta) &= \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{x_i}}{x_i!} = \frac{e^{-n\theta}\theta^{\sum x_i}}{\prod(x_i!)} \end{aligned}$$

Solution:

$$\begin{aligned} \text{Posterior: } \xi(\theta|\underline{x}) &\propto \frac{\theta e^{-10\theta}e^{-n\theta}\theta^{\sum x_i}}{\Gamma(2)10^2\prod(x_i!)} \\ &\propto e^{-\theta(n+10)}\theta^{\sum x_i+2-1} \\ \xi(\theta|\underline{x}) &= \frac{(n+10)^{\sum x_i+2}}{\Gamma(\sum x_i+2)}e^{-\theta(n+10)}\theta^{\sum x_i+2-1} \\ \theta|\underline{x} &\sim \text{Gamma}\left(\sum x_i + 2 = 6, n+10 = 22\right) \\ \delta^*(\underline{X}) &= \frac{\sum X_i + 2}{n+10} \\ \delta^*(\underline{x}) &= \frac{6}{22} = \frac{3}{11} \end{aligned}$$

Note: The Gamma distribution is parameterized slightly differently in DeGroot and on the sheet linked from my website (as is the exponential). Make sure the expected value matches what you've been given in the problem.