Math 152, Fall 2022
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WU # 15
Tuesday 11/1/22

Your name: ________________________________

Names of people you worked with: ________________________________

Task: Let $X \sim \text{Bin}\ (n, \theta)$. Consider the following estimator (called Agresti-Coull estimate, which has some similarity to the Bayesian estimator with a uniform prior but was derived using non-Bayesian methods):

$$\hat{\theta} = T = \frac{X+2}{n+4}$$

Find the Cramér-Rao lower bound (CRLB) for the variance of T.

Solution: The CRLB says:

$$var(T) \geq \frac{[m'(\theta)]^2}{nI(\theta)}$$

$$\begin{aligned}
m(\theta) &= E(T) = E\left[\frac{X+2}{n+4}\right] \\
&= \frac{n\theta+2}{n+4} \\
m'(\theta) &= \frac{n}{n+4} \\
(m'(\theta))^2 &= \left(\frac{n}{n+4}\right)^2 \\
\lambda(\theta) &= \ln\left(\binom{n}{X}\right) + X\ln(\theta) + (n-X)\ln(1-\theta) \\
\lambda'(\theta) &= \frac{X}{\theta} - \frac{(n-X)}{1-\theta} = \frac{X}{\theta(1-\theta)} \\
I_n(\theta) &= Var(\lambda'(\theta)) \\
&= \frac{Var(X)}{\theta^2(1-\theta)^2} = \frac{n\theta(1-\theta)}{\theta^2(1-\theta)^2} \\
&= \frac{n}{\theta(1-\theta)}
\end{aligned}$$

Which tells us that the variance of T is bounded by:

$$Var(T) \geq \frac{n}{(n+4)^2} \cdot \theta(1-\theta)$$

Turns out (some of you may have already noticed this), that we can calculate the variance of T directly!

$$Var(T) = \frac{Var(X)}{(n+4)^2} = \frac{n\theta(1-\theta)}{(n+4)^2}$$

Which is to say, the Agresti-Coull estimator reaches the CRLB. There is no other estimator with the same expected value that has a lower variance.