Math 152
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Bayesian Dist of the Mean

Theorem: Let $X_1, X_2, \dots X_n \sim N(\mu, 1/\tau)$ and suppose you have **priors** on $\mu|\tau$ and τ,

$$\begin{aligned} \mu|\tau &\sim N(\mu_0, 1/(\lambda_0\tau)) \\ \tau &\sim \text{Gamma}(\alpha_0, \beta_0) \end{aligned}$$

Then, the **posteriors** on $\mu|\tau$ and τ are,

$$\begin{aligned} \mu|\tau, \underline{x} &\sim N(\mu_1, 1/(\lambda_1\tau)) \\ \tau \mid \underline{x} &\sim \text{Gamma}(\alpha_1, \beta_1) \end{aligned}$$

where $\mu_1 = \frac{\lambda_0\mu_0 + n\overline{x}}{\lambda_0 + n}$, $\quad \lambda_1 = \lambda_0 + n$, $\quad \alpha_1 = \alpha_0 + \frac{n}{2}$, $\quad \beta_1 = \beta_0 + \frac{1}{2}\sum_{i=1}^n (x_i - \overline{x})^2 + \frac{n\lambda_0(\overline{x}-\mu_0)^2}{2(\lambda_0+n)}$.

Proof:

$$\begin{aligned} f(\underline{x}|\mu,\tau) &= \left(\frac{\tau}{2\pi}\right)^{n/2} exp\left[-\frac{1}{2}\tau\sum_{i=1}^n (x_i - \mu)^2\right] \\ \xi_1(\mu|\tau) &= \left(\frac{\lambda_0\tau}{2\pi}\right)^{1/2} exp\left[-\frac{1}{2}\lambda_0\tau(\mu-\mu_0)^2\right] \\ \xi_2(\tau) &= \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)}\tau^{\alpha_0-1}e^{-\beta_0\tau} \\ \text{Note,} &\quad \mu \text{ and } \tau \text{ are } \textbf{not} \text{ independent, and } \xi(\mu,\tau) = \xi_1(\mu|\tau)\ \xi_2(\tau) \end{aligned}$$

$$\begin{aligned} \xi(\mu,\tau|\underline{x}) &\propto f(\underline{x}|\mu,\tau)\ \xi_1(\mu|\tau)\ \xi_2(\tau) \\ &\propto \tau^{\alpha_0+(n+1)/2-1}\exp\left[-\frac{\tau}{2}\left(\lambda_0[\mu-\mu_0]^2 + \sum_{i=1}^n (x_i-\mu)^2\right) - \beta_0\tau\right] \qquad (1) \end{aligned}$$

As seen previously, we can add and subtract $\overline{x}$ inside $(x_i - \mu)^2$ to get:

$$\sum_{i=1}^n (x_i-\mu)^2 = \sum_{i=1}^n (x_i - \overline{x})^2 + n(\overline{x}-\mu)^2 \qquad (2)$$

By adding and subtracting μ_1:

$$n(\overline{x}-\mu)^2 + \lambda_0(\mu-\mu_0)^2 = (\lambda_0+n)(\mu-\mu_1)^2 + \frac{n\lambda_0(\overline{x}-\mu_0)^2}{\lambda_0+n} \qquad (3)$$

Combining (2) and (3) we get:

$$\sum_{i=1}^n (x_i-\mu)^2 + \lambda_0(\mu-\mu_0)^2 = (\lambda_0+n)(\mu-\mu_1)^2 + \sum_{i=1}^n (x_i-\overline{x})^2 + \frac{n\lambda_0(\overline{x}-\mu_0)^2}{\lambda_0+n} \qquad (4)$$

By plugging (4) into (1) we get:

$$\begin{aligned} \xi(\mu,\tau|\underline{x}) &\propto \left\{\tau^{1/2}exp\left[-\frac{1}{2}\lambda_1\tau(\mu-\mu_1)^2\right]\right\}(\tau^{\alpha_1-1}e^{-\beta_1\tau}) \qquad (5) \\ \xi(\mu,\tau|\underline{x}) &= \xi_1(\mu|\tau,\underline{x})\xi_2(\tau|\underline{x}) \qquad (6) \end{aligned}$$

$\square$

Theorem: Let $X_1, X_2, \ldots X_n \sim N(\mu, 1/\tau)$ and suppose you have priors on $\mu|\tau$ and τ,

$$\begin{aligned} \mu|\tau &\sim N(\mu_0, 1/(\lambda_0\tau)) \\ \tau &\sim \text{Gamma}(\alpha_0, \beta_0) \end{aligned}$$

Then, the marginal posterior distribution of μ can be written as:

$$\left(\frac{\lambda_1\alpha_1}{\beta_1}\right)^{1/2}(\mu - \mu_1) \mid \underline{x} \sim t_{2\alpha_1}$$

where $\mu_1, \lambda_1, \alpha_1$, and β_1 are given in the previous theorem.

Proof:

First, let

$$\begin{aligned} z &= (\lambda_1\tau)^{1/2}(\mu - \mu_1) = u(\mu) \\ \mu &= z(\lambda_1\tau)^{-1/2} + \mu_1 = w(z) \end{aligned}$$

We know (from the previous theorem):

$$\begin{aligned} \xi(\mu, \tau|\underline{x}) &= \xi_1(\mu|\tau, \underline{x}) \;\; \xi_2(\tau|\underline{x}) && (7) \\ \text{So, } \xi(z, \tau|\underline{x}) &= \xi_1(w(z)|\tau, \underline{x}) \;\; \left|\frac{\partial w(z)}{\partial z}\right| \;\; \xi_2(\tau|\underline{x}) && (8) \\ &= \xi_1(z(\lambda_1\tau)^{-1/2} + \mu_1|\tau, \underline{x}) \;\; |(\lambda_1\tau)^{-1/2}| \;\; \xi_2(\tau|\underline{x}) && (9) \\ &= \sqrt{\frac{\lambda_1\tau}{2\pi}}\exp\left\{\frac{-(z(\lambda_1\tau)^{-1/2} + \mu_1 - \mu_1)^2}{2(\lambda_1\tau)^{-1}}\right\} \;\; (\lambda_1\tau)^{-1/2} \;\; \xi_2(\tau|\underline{x}) && (10) \\ &= \sqrt{\frac{1}{2\pi}}\exp(-z^2/2) \;\; \xi_2(\tau|\underline{x}) && (11) \\ &= \Phi(z|\underline{x}) \;\; \xi_2(\tau|\underline{x}) && (12) \\ & && (13) \end{aligned}$$

Which gives us:

$$\begin{aligned} Z|\underline{x} &\sim N(0,1) \qquad \tau|\underline{x} \sim \text{Gamma}(\alpha_1, \beta_1) \quad \text{(Independent!)} \\ \text{Let } Y &= 2\beta_1\tau \rightarrow Y|\underline{x} \sim \text{Gamma}(\alpha_1, 1/2) \equiv \chi^2_{2\alpha_1} \end{aligned}$$

So, creating a t random variable:

$$U = \frac{Z}{\sqrt{Y/2\alpha_1}} = \frac{(\lambda_1\tau)^{1/2}(\mu - \mu_1)}{\sqrt{2\beta_1\tau/2\alpha_1}} = \left(\frac{\lambda_1\alpha_1}{\beta_1}\right)^{1/2}(\mu - \mu_1) \tag{14}$$

Which gives:

$$\left(\frac{\lambda_1\alpha_1}{\beta_1}\right)^{1/2}(\mu - \mu_1) \mid \underline{x} \sim t_{2\alpha_1} \tag{15}$$

$\square$